ELSEVIER
Journal of Geometry and Physics 26 (1998) 291–310
JOURNAL OF GEOMETRY AND PHYSICS

Bihamiltonian reductions and w_n-algebras

Paolo Casati [a,1], Gregorio Falqui [b,*], Franco Magri [a,2], Marco Pedroni [c,3]

[a] *Dipartimento di Matematica, Università di Milano, Via C. Saldini 50, I-20133 Milano, Italy*
[b] *SISSA, Via Beirut 2/4, I-34014 Trieste, Italy*
[c] *Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, I-16146 Genova, Italy*

Received 4 July 1997

Abstract

We discuss the geometry of the Marsden–Ratiu (MR) reduction theorem for a bihamiltonian manifold. We consider the case of the manifolds associated with the Gel'fand–Dickey theory, i.e., loop algebras over $\mathfrak{sl}_n$. We provide an explicit identification, tailored on the MR reduction, of the Adler–Gel'fand–Dickey brackets (AGD) with the Poisson brackets on the reduced bihamiltonian manifold $\mathcal{N}$. Such an identification relies on a suitable immersion of $T^*\mathcal{N}$ into the algebra of pseudodifferential operators connected to geometrical features of the theory of (classical) $\mathcal{W}_n$-algebras.

Subj. Class.: Differential geometry
1991 MSC: 58F05
Keywords: Bihamiltonian reductions; $\mathcal{W}_n$-algebras

1. Introduction

$\mathcal{W}$-algebras are algebras with *quadratic* commutation relations admitting the Virasoro Lie Algebra as a subalgebra. They have been the object of extensive study in the last few years, after the identification of such a structure (due to Zamolodchikov [34]) as the extended symmetry algebras of relevant models of two-dimensional (Quantum) Conformal Field Theory. It was soon understood that a physically meaningful family of such algebras could be obtained as quantum deformations of the Adler–Gel'fand–Dickey (AGD) bracket

* Corresponding author. E-mail: falqui@sissa.it.
[1] E-mail: casati@vmimat.mat.unimi.it.
[2] E-mail: magri@vmimat.mat.unimi.it.
[3] E–mail: pedroni@dima.unige.it.

PII S0393-0440(97)00060-0

well known from the classical theory of soliton equations [3,13,18,21,27]. Such a discovery prompted a remarkable amount of work aiming both at the classification of *all possible* $\mathcal{W}$-algebras, and the study of their representations and geometrical aspects (see, for an updated review, [5]). A particularly fruitful approach proved to be the Drinfel'd–Sokolov (DS) reduction scheme for Kac–Moody Lie algebras [17], for the wide variety of examples it embodies and the appealing mathematics behind it (see, e.g., [6,19,20,23,25].)

In this paper we want to discuss some of the geometrical aspects of the theory of (classical) $\mathcal{W}$-algebras related to the Hamiltonian approach to infinite-dimensional integrable systems. We set our study in the framework of the Marsden–Ratiu (MR) reduction scheme [30] for Poisson manifolds, extended in [10] to the case of bihamiltonian manifolds, whose application to the Gel'fand–Dickey hierarchies [24] can be found in [8,10].

The MR and the DS reduction schemes can be compared as follows, in relation to the theory of "Hamiltonian systems with symmetry" (see, e.g., [1]). The basic datum of the DS scheme is a Poisson action of a group G on Poisson manifold $\mathcal{M}$. The group defines a momentum map and a Hamiltonian reduction of the manifold $\mathcal{M}$ (the *Marsden–Weinstein* reduction). The reduction process considers a submanifold $\mathcal{S}_G$ of $\mathcal{M}$ (a level surface of the momentum map), a foliation $\mathcal{E}_G$ of $\mathcal{S}_G$ (the orbits of the little group), and the reduced phase space $\mathcal{N} = \mathcal{S}_G/\mathcal{E}_G$. On the other hand, in the geometric scheme of the bihamiltonian MR reduction the two steps are defined in terms of *two* compatible Poisson brackets on $\mathcal{M}$. The submanifold $\mathcal{S}$ is a symplectic leaf of the first Poisson bracket, and a foliation $\mathcal{E}$ is generated on it by the restriction to $T\mathcal{S}$ of the image, via the second bracket, of the Casimir functions of the first one. The resulting quotient space $\mathcal{N} = \mathcal{S}/\mathcal{E}$ is a bihamiltonian manifold.

The first point of this paper (Section 2) is to discuss in some detail the interplay of the manifolds $\mathcal{M}$, $\mathcal{S}$, and $\mathcal{N}$ together with their Poisson structures. We pay special attention to the Poisson brackets *on 1-forms*. This fits the results of [25] that identify classical $\mathcal{W}_n$ algebras as Poisson algebras of 1-forms over GD manifolds of monic differential operators of order $n+1$.

From Section 3 onwards, we specialize our constructions to the case of loop algebras $L(\mathfrak{sl}_n)$ over the simple Lie algebra $\mathfrak{g} = \mathfrak{sl}_n$, which we equip with the compatible Poisson structures:

$$\begin{aligned} &P_0 := \text{the commutator with a fixed element } A \in L(\mathfrak{sl}_n);\\ &P_1 := \text{the modified Kirillov–Kostant one.} \end{aligned} \tag{1.1}$$

In Sections 4 and 5 we identify the reduced Poisson brackets with the well-known linear and quadratic AGD brackets on the algebra of pseudodifferential operators (see, e.g., [2,14]), and provide the concrete examples of the KdV and Boussinesq cases.

It is worthwhile to remark that there are several ways to provide the phase space of the Gel'fand–Dickey theories a (bi)hamiltonian structure, and namely:

(1) the AGD procedure [14,29];
(2) the DS reduction from the space of matrix-valued first-order differential operators;
(3) the bihamiltonian reduction process on a loop algebra.

The equivalence between (1) and (2) is one of the results of DS seminal paper [17], while the one between (3) and (2) has been proved in [32], using the fact that the Marsden–Weinstein symmetry reduction can be seen as a particular case of the MR reduction. This paper provides a direct and *constructive* proof of the equivalence between the AGD brackets and those obtained by Hamiltonian reduction in the MR framework. Its main aim is the description of the geometrical properties of the MR reduction processes. Accordingly, we explicitly construct the embedding of $T^*\mathcal{N}$ into the algebra of pseudodifferential operators which allows us to perform the identification between the MR-reduced Poisson brackets and the AGD ones. In particular, we make contact with the results of [33] (see also [15]) about the factorization of the AGD quadratic Poisson tensor into a pair of Poisson algebra homomorphisms. We show indeed that the latter are a particular instance of the Poisson morphisms discussed in Section 2. Since each of those mappings corresponds to a precise step in the MR reduction process, our results may provide the theory of the $\mathcal{W}_n$-algebras some further geometrical flavor.

2. Marsden–Ratiu (MR) reduction

In this section we will briefly describe the bihamiltonian reduction process developed in [10], which is based on the MR reduction theorem [30]. Let us first recall the few notions of the general theory of Poisson and bihamiltonian manifolds which are needed for our purposes.

A *Poisson manifold* is a manifold $\mathcal{M}$ endowed with a Poisson bracket $\{\cdot,\cdot\}$, i.e., a bilinear skewsymmetric composition law of C^∞-functions fulfilling the Leibniz rule and the Jacobi identity. The corresponding *Poisson tensor* P is the bivector field P on $\mathcal{M}$, considered as a linear skewsymmetric map $P : T^*\mathcal{M} \to T\mathcal{M}$, defined by

$$\{f, g\} = \langle \mathrm{d}f, P\,\mathrm{d}g\rangle. \tag{2.1}$$

Any Poisson bracket on functions induces a Poisson bracket on forms [16,28]. If α_1 and α_2 are arbitrary 1-forms on a Poisson manifold $\mathcal{M}$ and P is the Poisson tensor, the bracket $\{\alpha_1, \alpha_2\}_P$ is defined by its value on a vector field X by

$$\langle\{\alpha_1, \alpha_2\}_P, X\rangle = L_{P\alpha_1}\langle\alpha_2, X\rangle - L_{P\alpha_2}\langle\alpha_1, X\rangle - \langle\alpha_1, L_X(P)\alpha_2\rangle, \tag{2.2}$$

where $L_{P\alpha_1}\langle\alpha_2, X\rangle$ denotes the Lie derivative of the scalar function $\langle\alpha_2, X\rangle$ along the vector field $P\alpha_1$, and $L_X(P)$ is the Lie derivative of the Poisson tensor P along X.

A *bihamiltonian manifold* is a manifold endowed with a pair of compatible Poisson brackets $\{\cdot,\cdot\}_0$ and $\{\cdot,\cdot\}_1$. Two Poisson brackets are compatible if the linear combinations

$$\{f, g\}_\lambda := \{f, g\}_1 + \lambda\{f, g\}_0 \tag{2.3}$$

verify the Jacobi identity for any value of the parameter λ. This is tantamount to requiring that the cyclic compatibility condition

$$\begin{aligned}&\{f, \{g, h\}_0\}_1 + \{h, \{f, g\}_0\}_1 + \{g, \{h, f\}_0\}_1\\ &\quad+\{f, \{g, h\}_1\}_0 + \{h, \{f, g\}_1\}_0 + \{g, \{h, f\}_1\}_0 = 0\end{aligned} \tag{2.4}$$

holds for any triple of functions (f, g, h) on $\mathcal{M}$. In this case, the bracket $\{\cdot,\cdot\}_\lambda$ is called the *Poisson pencil* on $\mathcal{M}$ defined by $\{\cdot,\cdot\}_0$ and $\{\cdot,\cdot\}_1$.

In order to describe the bihamiltonian reduction process, let us recall that every Poisson manifold $(\mathcal{M}, P)$ is foliated in symplectic leaves. Indeed, the characteristic distribution $C = \{P\,\mathrm{d}f \mid f \in C^\infty(\mathcal{M})\}$ is integrable, and the maximal integral leaves are symplectic manifolds [26]. On a bihamiltonian manifold the symplectic leaves of both Poisson tensors can be further foliated. Let us denote by C the characteristic distribution of the Poisson tensor P_0. A second distribution D, defined by $D = \{P_1\,\mathrm{d}f \mid f$ is a Casimir function of $P_0\}$, is naturally conjugated to C. The reduction theory of bihamiltonian manifolds is the study of the interplay between these two distributions. As a consequence of the compatibility condition (2.4) between the Poisson brackets, the distribution D is integrable [10]. Let us choose a specific symplectic leaf $\mathcal{S}$ of $\{\cdot,\cdot\}_0$, and let us denote by E the distribution induced on $\mathcal{S}$ by D; thus the leaves of E are the intersections of $\mathcal{S}$ with the leaves of D. We shall assume E sufficiently regular that there exists the quotient space $\mathcal{N} = \mathcal{S}/E$, and denote by $i_\mathcal{S} : \mathcal{S} \to \mathcal{M}$ and $\pi : \mathcal{S} \to \mathcal{N}$ the canonical immersion of $\mathcal{S}$ in $\mathcal{M}$ and the canonical projection of $\mathcal{S}$ onto $\mathcal{N}$. Then [10]:

Proposition 2.1. *The quotient space $\mathcal{N} = \mathcal{S}/E$ is a bihamiltonian manifold. On $\mathcal{N}$ there exists a unique Poisson pencil $\{\cdot,\cdot\}_\lambda^\mathcal{N}$ such that*

$$\{f, g\}_\lambda^\mathcal{N} \circ \pi = \{F, G\}_\lambda \circ i_\mathcal{S} \tag{2.5}$$

for any pair of functions F and G which extend the functions f and g of $\mathcal{N}$ into $\mathcal{M}$, and are constant on D. Technically, this means that the function F verifies the conditions

$$F \circ i_\mathcal{S} = f \circ \pi, \tag{2.6}$$

$$\{F, K\}_1 = 0 \tag{2.7}$$

for any function K whose differential, at the points of $\mathcal{S}$, belongs to the kernel of P_0.

Since in this paper we will deal with Poisson tensors rather than brackets, it is worthwhile to discuss the meaning of the previous proposition in terms of Poisson tensors. First of all we prove the following.

Lemma 2.2. *Let $s \in \mathcal{S}$ and $v \in T_s^*\mathcal{M}$. Then v is in the annihilator $(D^0)_s$ of D at s if and only if $(P_\lambda)_s v$ is tangent to $\mathcal{S}$.*

Proof. $(P_\lambda)_s v$ is tangent to $\mathcal{S}$ if and only if $\langle w, (P_\lambda)_s v\rangle = 0$ for all $w \in \mathrm{Ker}(P_0)_s$. But this is equivalent to the statement that $\langle v, (P_1)_s w\rangle = 0$ for all $w \in \mathrm{Ker}(P_0)_s$, i.e., that $v \in (D^0)_s$. □

To construct the reduced Poisson pencil $P_\lambda^\mathcal{N}$ starting from the Poisson pencil P_λ on $\mathcal{M}$ we have to conform to the following scheme:

(1) For any 1-form α on $\mathcal{N}$, and we consider the 1-form $\pi^*\alpha$ on $\mathcal{S}$, which obviously belongs to the annihilator E^0 of E in $T^*\mathcal{S}$ since $E = \mathrm{Ker}\,\pi_*$.

(2) We construct a *lifting* of α, that is, a 1-form β on $\mathcal{M}$ which belongs to the annihilator D^0 of D and satisfies

$$i_S^* \beta = \pi^* \alpha. \tag{2.8}$$

Such a lifting β of α is not uniquely defined, but this arbitrariness is irrelevant.

(3) We construct the vector field $P_\lambda \beta$ associated with the 1-form β through the Poisson pencil of $\mathcal{M}$. Thanks to Lemma 2.2, we have that $P_\lambda \beta$ is tangent to $\mathcal{S}$.

(4) Finally, we project this vector field from $\mathcal{S}$ to $\mathcal{N}$. The projection of $P_\lambda \beta$ does not depend on the choice of the particular lifting β and defines unambiguously $(P_\lambda^{\mathcal{N}})\alpha$.

We notice that in the construction of the reduced pencil $P_\lambda^{\mathcal{N}}$ only the value of the 1-form β at the points of $\mathcal{S}$ plays a role. Moreover, even if $\mathcal{S}$ is *not* a bihamiltonian manifold, to any lifting β of a 1-form on $\mathcal{N}$, we can associate a whole pencil of vector fields $X_\lambda = P_\lambda(\beta)$ *tangent* to $\mathcal{S}$.

Let us denote by $\mathcal{X}(\mathcal{M})$ (resp. $\mathcal{X}^*(\mathcal{M})$) the space of vector fields (resp. 1-forms) on $\mathcal{M}$, by $\Gamma(\mathcal{V})$ the space of sections of the bundle $\mathcal{V} \to \mathcal{M}$, and by $i_S^*(\mathcal{V})$ its pull-back on $\mathcal{S}$. In the space of sections of $i_S^*(T^*\mathcal{M})$ we can define the subspace Γ_S formed by all sections of $i_S^*(D^0)$ whose restriction to $\mathcal{X}(\mathcal{S})$ are liftings of 1-forms on $\mathcal{N}$:

$$\Gamma_S = \{\beta \in \Gamma(i_S^*(D^0)) |\ \beta_{|\mathcal{X}(\mathcal{S})} \in \operatorname{Im} \pi^*\}. \tag{2.9}$$

By definition of Γ_S, a surjective map $J_S : \Gamma_S \to \mathcal{X}^*(\mathcal{N})$ can be introduced by means of

$$\pi^*(J_S(\beta)) = i_S^*(\beta). \tag{2.10}$$

Then the previous scheme about the definition of the reduced Poisson pencil can be summarized in the formula

$$P_\lambda^{\mathcal{N}} \circ J_S = \pi_* \circ P_\lambda. \tag{2.11}$$

Proposition 2.3. *The space Γ_S is closed with respect to the Poisson brackets on 1-forms $\{\cdot,\cdot\}_{P_\lambda}$. Moreover the map J_S is a morphism between the Lie algebras $(\Gamma_S, \{\cdot,\cdot\}_{P_\lambda})$ and $(\mathcal{X}^*(\mathcal{N}), \{\cdot,\cdot\}_{P_\lambda^{\mathcal{N}}})$.*

Proof. First of all we remark that, since for all $\beta \in \Gamma_S$ the vector field $P_\lambda \beta$ is tangent to $\mathcal{S}$, the right-hand side of the expression

$$\langle \{\beta_1, \beta_2\}_{P_\lambda}, X\rangle = L_{P_\lambda \beta_1}\langle \beta_2, X\rangle - L_{P_\lambda \beta_2}\langle \beta_1, X\rangle - \langle \beta_1, L_X(P_\lambda)\beta_2\rangle, \tag{2.12}$$

where $\beta_1, \beta_2 \in \Gamma_S$, defines a section of $i_S^*(T^*\mathcal{M})$. In order to prove that $\{\beta_1, \beta_2\}_{P_\lambda} \in \Gamma_S$ we have first to check that, for all Y in D,

$$\langle \{\beta_1, \beta_2\}_{P_\lambda}, Y\rangle = 0. \tag{2.13}$$

Since D is generated by the Hamiltonian vector fields $P_1\,\mathrm{d}f$ with Hamiltonians f which are Casimirs of P_0, we can consider $Y = P_1\,\mathrm{d}f$ with $P_0\,\mathrm{d}f = 0$. Being $\beta_1, \beta_2 \in \Gamma_S$, we have $L_{P_\lambda \beta_2}\langle \beta_1, Y\rangle = L_{P_\lambda \beta_1}\langle \beta_2, Y\rangle = 0$; hence we simply have to prove that $\langle \beta_1, L_Y(P_\lambda)\beta_2\rangle$

$= 0$. Actually this is a consequence of the following formula (equivalent to the compatibility condition (2.4)):

$$L_{P_1\,\mathrm{d}f}(P_0) + L_{P_0\,\mathrm{d}f}(P_1) = 0 \quad \forall f \in C^\infty(\mathcal{M}). \tag{2.14}$$

We are thus left with showing that the Poisson bracket $\{\beta_1, \beta_2\}_{P_\lambda}$ restricted to $\mathcal{X}(\mathcal{S})$ is in the image of π^*. We will show both this property and that $J_{\mathcal{S}}$ is a morphism by proving that

$$\langle\{\beta_1, \beta_2\}_{P_\lambda}, X\rangle = \langle\{J_{\mathcal{S}}(\beta_1), J_{\mathcal{S}}(\beta_2)\}_{P_\lambda^{\mathcal{N}}}, \pi_*(X)\rangle \tag{2.15}$$

holds true for any vector field X tangent to $\mathcal{S}$ and projectable onto $\mathcal{N}$. From (2.12) and the identity $\langle\beta_1, L_X(P_\lambda)\beta_2\rangle\rangle = \langle J_{\mathcal{S}}(\beta_1), L_{\pi_*(X)}(P_\lambda^{\mathcal{N}})J_{\mathcal{S}}(\beta_2)\rangle$, which can be proved using properties of the Lie derivative and relation (2.11), we have:

$$\begin{aligned}\langle\{\beta_1, \beta_2\}_{P_\lambda}, X\rangle &= L_{P_\lambda\beta_1}\langle\beta_2, X\rangle - L_{P_\lambda\beta_2}\langle\beta_1, X\rangle - \langle\beta_1, L_X(P_\lambda)\beta_2\rangle \\ &= \frac{\partial}{\partial t_1}\langle J_{\mathcal{S}}(\beta_2), \pi_*(X)\rangle - \frac{\partial}{\partial t_2}\langle J_{\mathcal{S}}(\beta_1), \pi_*(X)\rangle \\ &\quad - \langle J_{\mathcal{S}}(\beta_1), L_{\pi_*(X)}(P_\lambda^{\mathcal{N}})J_{\mathcal{S}}(\beta_2)\rangle \\ &= \langle\{J_{\mathcal{S}}(\beta_1), J_{\mathcal{S}}(\beta_2)\}_{P_\lambda^{\mathcal{N}}}, \pi_*(X)\rangle,\end{aligned} \tag{2.16}$$

where we have denoted the Lie derivative along $P_\lambda^{\mathcal{N}}(J_{\mathcal{S}}(\beta_i))$ by $\partial/\partial t_i$. □

2.1. The transversal submanifold

In this section we recall the technique of the *transversal submanifold* described in [11]. Besides allowing to simplify the calculation involved in the bihamiltonian reduction scheme, it will naturally give rise to another space of sections $\Gamma_{\mathcal{Q}}$ which is closed with respect the Poisson bracket on 1-forms, and to a Poisson morphism $J_{\mathcal{Q}}: \Gamma_{\mathcal{Q}} \to \mathcal{X}^*(\mathcal{N})$ which, under an additional assumption, becomes an isomorphism.

In the notations of the previous section, a transversal submanifold to the distribution E is a submanifold $\mathcal{Q}$ of $\mathcal{S}$, which intersects every integral leaves of the distribution E in one and only one point. This condition implies the following relation on the tangent spaces:

$$T_Q\mathcal{S} = T_Q\mathcal{Q} \oplus E_q \quad \forall Q \in \mathcal{Q}. \tag{2.17}$$

If such a transversal submanifold $\mathcal{Q}$ exists, then it is obviously diffeomorphic to the quotient manifold $\mathcal{N}$, and inherits from $\mathcal{N}$ a bihamiltonian structure. The Poisson pencil on $\mathcal{N}$ can be computed by noticing [11] that given a 1-form $\alpha \in \mathcal{X}^*(\mathcal{N})$ there always exists a section β of $i_{\mathcal{Q}}^*(T^*\mathcal{M})$ where $i_{\mathcal{Q}} : \mathcal{Q} \to \mathcal{M}$ is the canonical immersion, such that

$$P_\lambda\beta \in \mathcal{X}(\mathcal{Q}), \tag{2.18}$$

$$\langle\beta, Y\rangle = \langle\alpha, \pi_*^{\mathcal{Q}}Y\rangle \quad \forall Y \in \mathcal{X}(\mathcal{Q}), \tag{2.19}$$

where $\pi^{\mathcal{Q}}$ is the projection from $\mathcal{Q}$ to $\mathcal{N}$, i.e., the restriction of π to $\mathcal{Q}$. Therefore in order to compute the action of the reduced Poisson pencil $P_\lambda^{\mathcal{N}}$ on the 1-form α, one has simply to determine the expression of $P_\lambda\beta$.

The sections β satisfying (2.18) form a subset $\Gamma_{\mathcal{Q}}$ of sections of $i^*_{\mathcal{Q}}(T^*\mathcal{M})$, and a natural map $J_{\mathcal{Q}} : \Gamma_{\mathcal{Q}} \to \mathcal{X}^*(\mathcal{N})$ is defined. Moreover, if $\beta \in \Gamma_{\mathcal{Q}}$, then the value β_Q of β at Q belongs to $(D^0)_Q$ for all $Q \in \mathcal{Q}$ (again by Lemma 2.2). From (2.2) it can be seen that the bracket $\{\beta_1, \beta_2\}_{P_\lambda}$ of two elements of $\Gamma_{\mathcal{Q}}$ is a well-defined element of $\Gamma_{\mathcal{Q}}$. Indeed, to check this, it is enough to show that $P_\lambda\{\beta_1, \beta_2\}_{P_\lambda} \in \mathcal{X}(\mathcal{Q})$. Since $\mathcal{Q}$ is a submanifold, this follows from the relation (see [16]):

$$P_\lambda\{\beta_1, \beta_2\}_{P_\lambda} = [P_\lambda\beta_1, P_\lambda\beta_2]. \tag{2.20}$$

In the same way it can be proved that the map $J_{\mathcal{Q}}$ owns the same properties of the map J_S, i.e., $J_{\mathcal{Q}}$ is a Lie algebra morphism from $\Gamma_{\mathcal{Q}}$ to $\mathcal{X}^*(\mathcal{N})$ and satisfies the relation

$$\pi_*^{\mathcal{Q}} \circ P_\lambda = P_\lambda^{\mathcal{N}} \circ J_{\mathcal{Q}}. \tag{2.21}$$

Finally we observe that if the kernels of the Poisson tensors P_0 and P_1 have trivial intersection on $\mathcal{Q}$, then the map $J_{\mathcal{Q}}$ becomes a Poisson isomorphism. It holds, indeed, the following

Proposition 2.4. *If* $\mathrm{Ker}\,(P_0) \cap \mathrm{Ker}\,(P_1) = \{0\}$ *on* $\mathcal{Q}$, *then for all* $\alpha \in \mathcal{X}^*(\mathcal{N})$ *there exists a unique lifting* $\beta \in \Gamma_{\mathcal{Q}}$ *satisfying conditions* (2.18)–(2.19). *Therefore* $J_{\mathcal{Q}}$ *is an isomorphism.*

Proof. We have only to prove that $J_{\mathcal{Q}}$ is injective. Let us therefore suppose that there exist β_1 and β_2 in $\Gamma_{\mathcal{Q}}$ such that $J_{\mathcal{Q}}(\beta_1) = J_{\mathcal{Q}}(\beta_2)$. Then, using Eq. (2.21), $\pi_*(P_\lambda(\beta_1 - \beta_2)) = P_\lambda^{\mathcal{N}}(J_{\mathcal{Q}}(\beta_1 - \beta_2)) = 0$. Since $P_\lambda(\beta_1 - \beta_2) \in \mathcal{X}(\mathcal{Q})$ and $\mathcal{Q}$ is a transversal submanifold we have that $P_\lambda(\beta_1 - \beta_2) = 0\,\forall\lambda$. This implies $\beta_1 - \beta_2 \in \mathrm{Ker}(P_0) \cap \mathrm{Ker}(P_1) = \{0\}$. □

3. Gel'fand–Dickey manifolds

In this section we will specialize the MR reduction scheme to the class of bihamiltonian manifolds which correspond to the Gel'fand–Dickey theories and their associated $\mathcal{W}_n$-algebras [17,18,20,25]. Let $\mathfrak{g}$ be the simple Lie algebra $\mathfrak{sl}(n+1)$ and $\mathcal{M} = \mathfrak{G}$ the space of C^∞-maps from S^1 into $\mathfrak{g}$. We denote by x the coordinate on S^1, and by S a map from S^1 into $\mathfrak{g}$. An element in the tangent space $T_S\mathcal{M}$ is denoted by $\dot{S}$. Identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ by means of the Killing form $(\cdot, \cdot)$, a covector is a map V from S^1 into $\mathfrak{g}$, whose value on the tangent vector $\dot{S}$ is given by

$$\langle V, \dot{S}\rangle = \int_{S^1} (V(x), \dot{S}(x))\,\mathrm{d}x. \tag{3.1}$$

The first Poisson bracket on $\mathcal{M}$ is defined by

$$\{f, g\}_0 = -\langle A, [\mathrm{d}f(S), \mathrm{d}g(S)]\rangle, \tag{3.2}$$

where f and g are arbitrary functionals on $\mathcal{M}$, $\mathrm{d}f(S)$ and $\mathrm{d}g(S)$ are their differentials and A is the vector of minimal weight for the "usual" Cartan decomposition of $\mathfrak{g}$, i.e.,

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}. \tag{3.3}$$

The second Poisson bracket is defined by

$$\{f, g\}_1 = \sigma(\mathrm{d}f, \mathrm{d}g) + \langle S, [\mathrm{d}f(S), \mathrm{d}g(S)]\rangle, \tag{3.4}$$

where

$$\sigma(\dot{S}_1, \dot{S}_2) = \int_{S^1} \left(\dot{S}_1, \frac{\mathrm{d}}{\mathrm{d}x}\dot{S}_2\right) \mathrm{d}x \tag{3.5}$$

is the nontrivial cocycle on $\mathcal{M}$. The corresponding Poisson tensors are given by

$$(P_0)_S V = [A, V], \tag{3.6}$$

$$(P_1)_S V = V_x + [V, S], \tag{3.7}$$

where V_x denotes the derivative of the loop V with respect to x. The *Lie–Poisson pencil* P_λ is

$$(P_\lambda)_S V = V_x + [V, S + \lambda A]. \tag{3.8}$$

It is a standard result [26] that these brackets are compatible.

The reduction process starts with the choice of a specific symplectic leaf $\mathcal{S}$ of the Poisson tensor (3.6). All leaves are affine hyperplanes modeled on the orthogonal space $\mathfrak{G}_A^{\perp}$ (with respect to the pairing (3.1)) of $\mathfrak{G}_A = \{V \in \mathfrak{G} \text{ s.t. } [V, A] = 0\}$, and we choose the one passing through the sum B of the root vectors corresponding to the positive simple roots of $\mathfrak{g}$. Thus $S \in \mathcal{S}$ is parametrized by $2n$ periodic functions (p_a, q_a) as follows:

$$S = \begin{pmatrix} p_0 & 1 & \dots & \dots & 0 \\ p_1 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ p_{n-1} & \dots & \dots & \dots & 1 \\ q_0 & \dots & \dots & q_{n-1} & -p_0 \end{pmatrix}. \tag{3.9}$$

The Gel'fand–Dickey theories are related to the particular choice of the pair (A, B) above made but it is possible to make different choices of such elements. In [9] were considered pairs corresponding to the "fractional" (or generalized) KdV hierarchies [4,6] and in [31] those leading to the classical AKNS system.

The next step is the study of the foliation E. A basic feature of (generalized) GD theories is that, thanks to the form of the Kirillov–Kostant Poisson tensor (3.8), the integral leaves of E are orbits of a group action. From [7,11] we borrow:

Proposition 3.1. *The following properties hold:*

(i) *The distribution E is spanned by the vector fields*

$$X_V(S) = V_x + [V, S] \tag{3.10}$$

with $V \in \mathfrak{G}_{AB} = \{V \in \mathfrak{G}_A \mid V_x + [V, B] \in \mathfrak{G}_A^{\perp}\}$.

(ii) *The integral leaves of E are the orbits of the gauge action of $\mathcal{G}_{AB} = \exp(\mathfrak{G}_{AB})$ on $\mathcal{S}$ defined by*

$$S' = GSG^{-1} + G_x G^{-1}. \tag{3.11}$$

(iii) *The points of $\mathcal{S}$ with coordinates $p_j = 0$ form a submanifold $\mathcal{Q}$ of $\mathcal{S}$ transversal to the distribution E. The reduced bihamiltonian Gel'fand–Dickey manifold $\mathcal{N} = \mathcal{S}/E$ is therefore parametrized by n independent functions $(u_0, u_1, \dots, u_{n-1})$ on $\mathcal{S}$, invariant under $\mathcal{G}_{AB}$. The restriction to $\mathcal{Q}$ of the projection $\pi : \mathcal{S} \to \mathcal{N} = \mathcal{S}/E$ is simply given by the equations*

$$u_j = q_j \quad (j = 0, 1, \dots, n-1). \tag{3.12}$$

To compute the reduced Poisson pencil $P_\lambda^{\mathcal{N}}$ according to the general scheme discussed in Section 2, we will use the technique of the transversal manifold outlined in Section 2.1. It is worthwhile to remark that the points Q of $\mathcal{Q}$ have the canonical Frobenius form

$$Q = \begin{pmatrix} 0 & 1 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 1 \\ u_0 & u_1 & \dots & u_{n-1} & 0 \end{pmatrix}. \tag{3.13}$$

Proposition 3.2. *On the whole symplectic leaf $\mathcal{S}$ one has that* $\mathrm{Ker}(P_0) \cap \mathrm{Ker}(P_1) = \{0\}$.

Proof. The statement is proved recalling that the Lie algebra $\mathfrak{g}$ admits the gradation

$$\mathfrak{sl}_{n+1} = \bigoplus_{k=-n}^{n} \mathfrak{g}_k, \tag{3.14}$$

where $\mathfrak{g}_k$ is the space of matrices M such that $M_j^l = 0$ for $l - j \neq k$. This induces a corresponding grading in the loop algebra $\mathfrak{G}$. Let us now consider an element $V \in \mathrm{Ker}P_0$. Then its decomposition with respect to the gradation is

$$V = \sum_{k=-n}^{n-1} V_k \tag{3.15}$$

and each homogeneous component V_k is in $\mathfrak{G}_A$. Remark that any element of the symplectic leaf $\mathcal{S}$ can be decomposed as $S = T + B$ with $T \in \oplus_{k=-n}^{0} \mathfrak{g}_k$. Imposing that $V \in \mathrm{Ker}P_1$ and considering the maximal degree elements, we get $[B, V_{n-1}] = 0$. Hence, V_{n-1} commutes both with A and B. Then, since $\mathfrak{g}_A \cap \mathfrak{g}_B = \{0\}$ (see [11]), we can conclude that $V_{n-1} = 0$. In the same way, a recursive argument proves that $V = 0$. □

In particular, the statement of this proposition holds at the points of $\mathcal{Q}$, so that, recalling the discussion of Section 2.1, the morphism J_Q is an isomorphism, whose inverse will be denoted by ϕ.

The unique section $\mathbb{V} = \phi(v)$ in Γ_Q lifting the 1-form v in $\mathcal{X}^*(\mathcal{N})$ can be explicitly computed, using conditions (2.18) and (2.19). Let $v = (v_0, v_1, \dots, v_{n-1}) \in \mathcal{X}^*(\mathcal{N})$ be defined by $\langle v, \dot{u}\rangle = \sum_{i=0}^{n-1} \int_{S^1} v_i \dot{u}_i$ and let $(\mathbb{V}_i^j)_{i,j=0,\dots,n}$ be the entries of the matrix $\mathbb{V} = \phi(v)$; then (2.19) implies

$$\mathbb{V}_n^j = v_j, \quad j = 0, \dots, n-1. \tag{3.16}$$

Substituting the explicit expression of the Poisson pencil (3.8) in (2.18) we get

$$\mathbb{V}_x + [\mathbb{V}, Q + \lambda A] \in T_Q\mathcal{Q}. \tag{3.17}$$

This condition implies the following relations on the entries of the matrix $\mathbb{V}$:

$$\begin{array}{llllll}
-\mathbb{V}_0^{k+1} & & +\mathbb{V}_{0x}^k & + & \mathbb{V}_n^k(u_0+\lambda) & = 0, \\
-\mathbb{V}_1^{k+1}+ & \mathbb{V}_0^k + & \mathbb{V}_{1x}^k & + & \mathbb{V}_n^k u_1 & = 0, \\
-\mathbb{V}_2^{k+1}+ & \mathbb{V}_1^k + & \mathbb{V}_{2x}^k & + & \mathbb{V}_n^k u_2 & = 0, \\
\vdots & & \vdots & & \vdots & \\
-\mathbb{V}_n^{k+1} & +\mathbb{V}_{n-1}^k & + & \mathbb{V}_{nx}^k & & = 0.
\end{array} \tag{3.18}$$

These formulas together with the zero trace condition $\sum_{k=0}^{n} \mathbb{V}_k^k = 0$ show that each of the elements $\mathbb{V}_l^k$ is obtained algebraically from the first n elements of the last column of $\mathbb{V}$. With the elements of Γ_Q at our disposal, the reduction of the Poisson pencil can be now completed quite easily. According to the general procedure we have to compute the vector field

$$\dot{Q} = \mathbb{V}_x + [\mathbb{V}, Q + \lambda A], \tag{3.19}$$

where $\mathbb{V} = \phi(v) \in \Gamma_Q$. By expanding Eq. (3.19) we obtain

$$\begin{aligned}
\dot{u}_0 &= (\mathbb{V}_0^n)_x + \mathbb{V}_n^n(u_0+\lambda) - \sum_{l=0}^{n-1} u_l \mathbb{V}_0^l - \lambda \mathbb{V}_0^0, \\
\dot{u}_j &= (\mathbb{V}_j^n)_x + \mathbb{V}_{j-1}^n + \mathbb{V}_n^n u_j - \sum_{l=0}^{n-1} u_l \mathbb{V}_j^l - \lambda \mathbb{V}_j^0, \quad j = 1, \dots n-1
\end{aligned} \tag{3.20}$$

and

$$0 = (\mathbb{V}_n^n)_x + \mathbb{V}_{n-1}^n - \sum_{l=0}^{n-1} u_l \mathbb{V}_n^l - \lambda \mathbb{V}_n^0. \tag{3.21}$$

Eq. (3.20) give the explicit form of the reduced Poisson pencil. In order to obtain a more compact formula, to be used in the following section, we define

$$\begin{aligned} \mathbb{V}_0^{n+1} &= \mathbb{V}_{0x}^n + \mathbb{V}_n^n(u_0+\lambda), \\ \mathbb{V}_l^{n+1} &= \mathbb{V}_{lx}^n + \mathbb{V}_{l-1}^n + u_l\mathbb{V}_n^n, \quad l = 1, \dots, n, \end{aligned} \tag{3.22}$$

where $u_n := 0$. Then the reduced Poisson pencil (3.20) can be written as

$$\dot{u}_j = \mathbb{V}_j^{n+1} - \sum_{l=0}^{n-1} u_l \mathbb{V}_j^l - \lambda \mathbb{V}_j^0, \quad l = 0, \dots, n-1. \tag{3.23}$$

4. Adler–Gel'fand–Dickey (AGD) brackets

In this section we perform the identification of the reduced brackets on $\mathcal{N}$ with the AGD brackets and discuss some Hamiltonianaspects of Gel'fand–Dickey theories and $\mathcal{W}_n$-algebras.

The usual setting for Gel'fand–Dickey theories can be briefly described as follows. One considers the space ΨDO of pseudodifferential operators on S^1 (see, e.g., [2,12,14,29] for a broader account on the subject), i.e., the space of formal Laurent series of the form

$$A = \sum_{i=-\infty}^{N} a_i(x)\partial_x^i. \tag{4.1}$$

It is an associative algebra under the product defined (on homogeneous elements $A_1 = a_{k_1}\partial_x^{k_1}$, $A_2 = a_{k_2}\partial_x^{k_2}$) as

$$A_1 A_2 = \sum_{n\geq 0} \binom{n}{k_1} a_1(\partial_x^k a_2) \cdot \partial_x^{k_2+k_1-n}. \tag{4.2}$$

Its associated Lie algebra admits a filtration (indexed by the integers) via the subspaces ΨDO_p formed by those operators of order at most p, and a trace form (the *Adler trace*) given by

$$\operatorname{tr}(A) = \oint \operatorname{res}_\partial A = \oint a_{-1}(x). \tag{4.3}$$

It is customary to denote by A_+ the strictly differential part of A and by $A_- := A - A_+$.

Let $\mathcal{L}_{n+1}$ be the space of order $n+1$ monic differential operators on the circle, parametrized by the $(n+1)$-tuple of functions $\{u_0, \dots, u_n\}$. Its tangent space $T\mathcal{L}_{n+1}$ can be naturally identified with the space D_n of differential operators of order n, and (via the Adler trace) its cotangent space $T^*\mathcal{L}_{n+1}$ with the quotient space $\Psi DO/\Psi DO_{-(n+2)}$ of pseudodifferential operators modulo those of degree less than $-(n+1)$. It is a classical result that $\mathcal{L}_{n+1}$ is endowed with a compatible pair of Poisson tensor, which are defined by

$$\mathcal{P}_0(L)\cdot X = [X, L]_+, \qquad \mathcal{P}_1(L)\cdot X = L(XL)_+ - (LX)_+L, \tag{4.4}$$

and are usually called, respectively, the Gardner–Zakharov–Faddeev and the Adler–Gel'fand –Dickey brackets or collectively AGD brackets. It is also a well-known fact (see, e.g., [14]) that these brackets restrict to the subspace $u_n = 0$, where $L = \partial^{n+1} - \sum_{j=0}^{n-1} u_j \partial^j$.

The connection of our previous results with this picture is established simply by translating into this language the two processes involved in the MR reduction:
(1) the lifting of 1-forms from $\mathcal{X}^*(\mathcal{N})$ in Γ_Q;
(2) the projection of the Hamiltonian vector fields from $\mathcal{S}$ to $\mathcal{N}$.
This can be done as follows. With any $u \in \mathcal{N}$, $\dot{u} \in \mathcal{X}(\mathcal{N})$, $v \in \mathcal{X}^*(\mathcal{N})$, and $\mathbb{V} \in \Gamma_Q$ we associate the operators $(L, \dot{L}, \xi(v), \psi(\mathbb{V}))$ defined by

$$\begin{aligned} L &:= \partial^{n+1} - \sum_{j=0}^{n-1} u_j \partial^j, & \dot{L} &:= -\sum_{j=0}^{n-1} \dot{u}_j \partial^j, \\ \xi(v) &:= -\sum_{j=0}^{n} \partial^{-j-1} [\phi(v)]_n^j, & \psi(\mathbb{V}) &:= -\sum_{j=0}^{n} \mathbb{V}_j^0 \partial^j. \end{aligned} \tag{4.5}$$

Notice that $\mathbb{V}_j^0$ are the elements of the first row of $\mathbb{V}$ and $[\phi(v)]_n^j$ are the elements of the last column of the image under ϕ of v. For simplicity of notation, and to anticipate the content of the next subsection, we put $X = \xi(v)$ and $E = \psi(\mathbb{V})$. Since $[\phi(v)]_n^j = v_j$, for $j = 0, \ldots, n-1$, we have

$$\int_{S^1} \mathrm{res}_\partial (X\dot{L})\, \mathrm{d}x = \langle v, \dot{u} \rangle, \tag{4.6}$$

and so we are allowed to consider (L, X) as the representative of the 1-form v in the space of pseudodifferential operators. Our task is to establish a link between E and X, corresponding to the relation between v and the matrix $\mathbb{V}$ constructed in the previous section, and a link between E and $\dot{L}$, corresponding to the relation between the 1-form $\mathbb{V}$ and the vector field $\dot{u}$ given by formula (3.23). The key to solve this problem is given by:

Lemma 4.1. *Let the coefficients $\mathbb{V}_l^j$ be defined, for $j = 0, \ldots, n+1$ and $l = 0, \ldots, n$, by Eqs.* (3.18) *and* (3.22). *Then*

$$\partial^j E = -\sum_{l=0}^{n} \mathbb{V}_l^j \partial^l + (\partial^j E L^{-1})_+ (L - \lambda), \quad j = 0, \ldots, n+1. \tag{4.7}$$

Proof. We prove by induction that

$$\partial^j E = -\sum_{l=0}^{n} \mathbb{V}_l^j \partial^l + R_j (L - \lambda), \tag{4.8}$$

where R_j is a purely differential operator. It is true for $j = 0$ because of the definition of E. Moreover

$$\partial^{j+1} E = \partial(\partial^j E) = -\sum_{l=0}^{n} (\mathbb{V}_{l\,x}^j \partial^l + \mathbb{V}_l^j \partial^{l+1}) + \partial R_j (L - \lambda)$$

$$
\begin{aligned}
&= -\sum_{l=0}^{n} (\mathbb{V}^j_{l\,x} + \mathbb{V}^j_{l-1})\partial^l - \mathbb{V}^j_n \partial^{n+1} + \partial R_j (L-\lambda) \\
&= -\sum_{l=0}^{n} (\mathbb{V}^j_{l\,x} + \mathbb{V}^j_{l-1})\partial^l - \mathbb{V}^j_n \left(L + \sum_{l=0}^{n-1} u_l \partial^l \right) + \partial R_j (L-\lambda) \\
&= -\sum_{l=0}^{n} (\mathbb{V}^j_{l\,x} + \mathbb{V}^j_{l-1} + u_l \mathbb{V}^j_n)\partial^l - \mathbb{V}^j_n L + \partial R_j (L-\lambda),
\end{aligned} \tag{4.9}
$$

where we put $u_n = 0$ and $\mathbb{V}^j_l = 0$ for $l < 0$. Now we use the recursion relations (3.18) and the definition of $\mathbb{V}^{n+1}_j$ to get

$$
\begin{aligned}
\partial^{j+1} E &= -\sum_{l=0}^{n} \mathbb{V}^{j+1}_l \partial^l + \lambda \mathbb{V}^j_n - \mathbb{V}^j_n L + \partial R_j (L-\lambda) \\
&= -\sum_{l=0}^{n} \mathbb{V}^{j+1}_l \partial^l + (\partial R_j - \mathbb{V}^j_n)(L-\lambda),
\end{aligned} \tag{4.10}
$$

proving (4.8). Finally we notice that this equation implies that

$$
R_j = [(\partial^j E)(L-\lambda)^{-1}]_+ = [(\partial^j E)(L^{-1} + \lambda L^{-2} + \cdots)]_+ = (\partial^j E L^{-1})_+ \tag{4.11}
$$

for $j = 0, \ldots, n+1$. □

We are now able to prove the main relations between the pseudodifferential operators associated with our geometrical objects. From now on we put $\mathbb{V} = \phi(v)$ and, consequently, $E = \psi(\phi(v))$.

Proposition 4.2.

(i) *The operators E and X are related by $E = (XL)_+$. This equation is the operator form of the "lifting of covectors" entering the MR reduction.*

(ii) *The operators E and $\mathrm{d}L/\mathrm{d}t_i$ associated with the i-th reduced structure are related by*

$$
\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}t_1} &= LE - (LEL^{-1})_+ L, \\
\frac{\mathrm{d}L}{\mathrm{d}t_0} &= E - (LEL^{-1})_+.
\end{aligned} \tag{4.12}
$$

These equations are the operator form of the projection of the Hamiltonian vector fields from $\mathcal{S}$ onto $\mathcal{N}$.

(iii) *The operator form of the reduced Poisson tensor on $\mathcal{N}$ is given by*

$$
\begin{aligned}
\frac{\mathrm{d}L}{\mathrm{d}t_1} &= L(XL)_+ - (LX)_+ L, \\
\frac{\mathrm{d}L}{\mathrm{d}t_0} &= (XL)_+ - (LX)_+ = [X, L]_+,
\end{aligned} \tag{4.13}
$$

i.e., the reduced brackets are the AGD brackets in the algebra of pseudodifferential operators.

Proof.

(i) It is easily seen that the first claim is equivalent to the assertion that $EL^{-1} = X + Z$, with $\deg Z = -n - 2$. The latter follows from Lemma 4.1, since

$$\operatorname{res}(\partial^j EL^{-1}) = \operatorname{res}\left(-\sum_{l=0}^{n} \mathbb{V}_l^j \partial^l L^{-1} + (\partial^j X)_+ - \lambda(\partial^j X)_+ L^{-1}\right), \tag{4.14}$$

and $\deg(\partial^j X)_+ L^{-1} \le -2$ for $j \le n$. Therefore $\operatorname{res}(\partial^j EL^{-1}) = -\operatorname{res}\sum_{l=0}^{n} \mathbb{V}_l^j \partial^l L^{-1} = -\operatorname{res}(\mathbb{V}_n^j \partial^n L^{-1}) = -\mathbb{V}_n^j$ for $j = 0, \dots, n$. We conclude that $EL^{-1} = -(\partial^{-1}\mathbb{V}_n^0 + \cdots + \partial^{-(n+1)}\mathbb{V}_n^n) + Z$, with $\deg Z = -n-2$. Finally, $\sum_{j=1}^{n}(-\partial^{-j-1}\mathbb{V}_n^j) = X$ from (3.16).

(ii) Let us set $\dot L = \mathrm{d}L/\mathrm{d}t_1 - \lambda\, \mathrm{d}L/\mathrm{d}t_0$; then $\dot L = -\sum_{l=0}^{n-1} \dot u_j \partial^j$, where $\dot u_j$ is the vector field associated with the Poisson pencil (3.23):

$$\dot u_j = \mathbb{V}_j^{n+1} - \sum_{l=0}^{n-1} u_l \mathbb{V}_j^l - \lambda \mathbb{V}_j^0. \tag{4.15}$$

Then

$$\begin{aligned}\dot L &= \sum_{j=0}^{n-1}\left(-\mathbb{V}_j^{n+1} + \sum_{l=0}^{n-1} u_l \mathbb{V}_j^l + \lambda \mathbb{V}_j^0\right)\partial^j \\ &= \sum_{j=0}^{n}\left(-\mathbb{V}_j^{n+1} + \sum_{l=0}^{n-1} u_l \mathbb{V}_j^l + \lambda \mathbb{V}_j^0\right)\partial^j,\end{aligned} \tag{4.16}$$

since Eqs. (3.21) and the definition (3.22) of $\mathbb{V}_n^{n+1}$ imply that $\mathbb{V}_n^{n+1} - \sum_{l=0}^{n-1} u_l \mathbb{V}_n^l - \lambda \mathbb{V}_n^0 = 0$. Therefore we have

$$\begin{aligned}\dot L &= -\sum_{j=0}^{n} \mathbb{V}_j^{n+1}\partial^j + \sum_{l=0}^{n-1} u_l \sum_{j=0}^{n} \mathbb{V}_j^l \partial^j + \lambda \sum_{j=0}^{n} \mathbb{V}_j^0 \partial^j \\ &= \partial^{n+1}E - (\partial^{n+1}EL^{-1})_+(L-\lambda) \\ &\quad + \sum_{l=0}^{n-1} u_l(-\partial^l E + (\partial^l EL^{-1})_+(L-\lambda)) - \lambda E \\ &= \left(\partial^{n+1} - \sum_{l=0}^{n-1} u_l \partial^l\right) E - \left[(\partial^{n+1}EL^{-1})_+ - \sum_{l=0}^{n-1} u_l (\partial^l EL^{-1})_+\right] L \\ &\quad - \lambda\left(E + \sum_{l=0}^{n-1} u_l(\partial^l EL^{-1})_+ - (\partial^{n+1}EL^{-1})_+\right) \\ &= LE + RL - \lambda(E - (LEL^{-1})_+),\end{aligned} \tag{4.17}$$

where R is a purely differential operator. Thus

$$\frac{\mathrm{d}L}{\mathrm{d}t_1} = LE + RL, \qquad \frac{\mathrm{d}L}{\mathrm{d}t_0} = -E + (LEL^{-1})_+ \tag{4.18}$$

and

$$\frac{\mathrm{d}L}{\mathrm{d}t_1}L^{-1} = LEL^{-1} + R \implies R = -(LEL^{-1})_+. \tag{4.19}$$

By this result, one easily gets

$$\frac{\mathrm{d}L}{\mathrm{d}t_1} = L(XL)_+ - (LX)_+L, \qquad \frac{\mathrm{d}L}{\mathrm{d}t_0} = (XL)_+ - (LX)_+ = [X, L]_+, \tag{4.20}$$

so that (iii) is also proved. □

4.1. On Radul's morphism

We are now in a position to analyze from a geometrical point of view some results of [33] (furtherly generalized in [22]). They can be described as follows.

Let $\mathcal{E}$ be the algebra of differential operators with coefficients in the space of differential polynomials in the u_i's and $\mathcal{E}_{n+1}$ its quotient with respect to the equivalence relation

$$E \sim F \quad \text{iff} \quad F = E + RL \text{ for some } R \in \mathcal{E}. \tag{4.21}$$

Moreover, let W_L be the map associating with every element E of $\mathcal{E}$ the differential operator

$$W_L(E) := LE - (LEL^{-1})_+L \equiv (LEL^{-1})_-L. \tag{4.22}$$

Finally, define $\Theta_L(E) := (EL^{-1})_-$ and $\Phi_L(X) := (XL)_+$.

Proposition 4.3. *The following properties hold* [33]*:*

(i) *W_L takes values in $T\mathcal{L}_{n+1}$, passes to the quotient $\mathcal{E}_{n+1} = \mathcal{E}/\sim$, and is an isomorphism of Lie algebras, provided one defines on $\mathcal{E}_{n+1}$ the commutator*

$$[E, F]_L := [E, F] + W_L(E)\cdot(F) - W_L(F)\cdot(E) \; mod \; \sim. \tag{4.23}$$

Here $W_L(Y)\cdot(F) = (\mathrm{d}/\mathrm{d}\epsilon)|_{\epsilon=0}F(L + \epsilon W_L(Y))$;

(ii) *the map W_L factorizes as*

$$W_L = \mathcal{P}_1 \cdot \Theta_L, \tag{4.24}$$

where $\mathcal{P}_1$ is the Poisson tensor defined in Eq. (4.4)*;*

(iii) *Θ_L is a Lie algebras isomorphism between $(\mathcal{E}_{n+1}, [\cdot,\cdot]_L)$ and $(\mathcal{X}^*(\mathcal{L}_{n+1}), \{\cdot,\cdot\}_{\mathcal{P}_1})$, where $\{\cdot,\cdot\}_{\mathcal{P}_1}$ is the Poisson bracket on 1-forms induced by $\mathcal{P}_1$.*

Let us regard $\mathcal{E}_{n+1}$ as a fiber bundle over $\mathcal{L}_{n+1}$, and the map ψ of Eq. (4.5) as a map from Γ_Q to $\Gamma(\mathcal{E}_{n+1})$. We endow the space Γ_Q with the Poisson bracket on 1-forms corresponding to the Poisson tensor P_1 of Eq. (3.7), which can be shown to have the form

$$\{V_1, V_2\}_1 = \frac{\partial V_2}{\partial t_1} - \frac{\partial V_1}{\partial t_2} - [V_1, V_2], \tag{4.25}$$

where $\partial/\partial t_i$ is the directional derivative along the vector field $P_1 V_i$ and $[\cdot,\cdot]$ is the matrix commutator.

Proposition 4.4. *The map ψ is a Lie algebra homomorphism between $(\Gamma_Q, \{\cdot,\cdot\}_1)$ and $(\mathcal{E}_{n+1}, [\cdot,\cdot]_L)$.*

Proof. Let $\mathbb{V}, \mathbb{W} \in \Gamma_Q$, and let us put $E = \psi(\mathbb{V}) = -\sum_{j=0}^{n} \mathbb{V}_j^0 \partial^j$ and $F = \psi(\mathbb{W}) = -\sum_{j=0}^{n} \mathbb{W}_j^0 \partial^j$; then from formula (4.25) we have

$$\psi(\{\mathbb{V}, \mathbb{W}\}_1) = \frac{\partial F}{\partial t_1} - \frac{\partial E}{\partial t_2} - \psi([\mathbb{V}, \mathbb{W}]), \tag{4.26}$$

where $\partial/\partial t_1$ and $\partial/\partial t_2$ are the directional derivatives along the vector fields $P_1\mathbb{V}$ and $P_1\mathbb{W}$, respectively. It follows from Proposition 4.2 that $P_1\mathbb{V}$ (resp. $P_1\mathbb{W}$) is, in terms of pseudodifferential operators, $W_L(E)$ (resp. $W_L(F)$). Therefore we have

$$\frac{\partial F}{\partial t_1} - \frac{\partial E}{\partial t_2} = W_L(E)(F) - W_L(F)(E). \tag{4.27}$$

Therefore it remains to prove that

$$-\psi([\mathbb{V}, \mathbb{W}]) = \sum_{l,j=0}^{n} (\mathbb{V}_j^0 \mathbb{W}_l^j - \mathbb{W}_j^0 \mathbb{V}_l^j)\partial^l \sim [E, F]. \tag{4.28}$$

Using formula (4.8) with $\lambda = 0$, we have

$$\begin{aligned} [E, F] &= EF - FE \\ &= -\sum_{j=0}^{n} \mathbb{V}_j^0 \partial^j F + \sum_{j=0}^{n} \mathbb{W}_j^0 \partial^j E \\ &= \sum_{l,j=0}^{n} (\mathbb{V}_j^0 \mathbb{W}_l^j - \mathbb{W}_j^0 \mathbb{V}_l^j)\partial^l + RL, \end{aligned} \tag{4.29}$$

where R is a differential operator. Hence (4.28) follows, and the proof is completed. □

The results of this section can be summarized in the following commutative diagram of Lie algebra morphism

$$\begin{array}{ccc} (\mathcal{E}_{n+1},[\cdot,\cdot]_L) & \xrightarrow{\quad\Theta_L\quad} & (\mathcal{X}^*(\mathcal{L}_{n+1}),\{\cdot,\cdot\}_{\mathcal{P}_1}) \\ \psi\uparrow & & \uparrow\xi \\ (\Gamma_Q,\{\cdot,\cdot\}_{P_1}) & \xrightarrow{\quad J_Q\quad} & (\mathcal{X}^*(\mathcal{N}),\{\cdot,\cdot\}_{P_1^{\mathcal{N}}}). \end{array} \tag{4.30}$$

Notice that ξ is indeed a morphism since it allowed us to identify the AGD brackets and the reduced brackets in Proposition 4.2. Hence, the fact that Θ_L is a morphism of Lie algebras can be read as a consequence of the general properties of J_Q discussed in Section 2.1.

5. Examples

This last section is devoted to exemplify the results of this paper in the cases of the KdV and Boussinesq hierarchies.

The KdV hierarchy is the GD theory associated with the loop algebra $L(\mathfrak{sl}(2))$ whose picture in the MR scheme has been given in [10]. As we have seen in Section 2.1, to compute the reduced Poisson pencil and the maps ξ and ψ it is enough to calculate the elements of the subspace Γ_Q, where the transversal manifold Q is parametrized by

$$Q = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}. \tag{5.1}$$

The lifting $\mathbb{V}$ in Γ_Q of the 1-form v of $\mathcal{X}^*(\mathcal{N})$ is determined by conditions (3.16) and (3.18), and has the form

$$\mathbb{V} = \begin{pmatrix} -\frac{1}{2}v_x & v \\ -\frac{1}{2}v_{xx} + (u+\lambda)v & \frac{1}{2}v_x \end{pmatrix}. \tag{5.2}$$

The reduced Poisson pencil is given by formula (3.20):

$$\dot{u} = -\tfrac{1}{2}v_{xxx} + 2(u+\lambda)v_x + u_x v. \tag{5.3}$$

The Poisson brackets on 1-forms is

$$\{v_1, v_2\}_{P_\lambda^{\mathcal{N}}} = \frac{\partial v_2}{\partial t_1} - \frac{\partial v_1}{\partial t_2} - [v_1, v_2]_{\mathcal{N}}, \tag{5.4}$$

where $\partial/\partial t_i$ is the directional derivative along the vector field $P_\lambda^{\mathcal{N}} v_i$, and

$$[v_1, v_2]_{\mathcal{N}} = v_1 v_{2x} - v_2 v_{1x}. \tag{5.5}$$

The Lax operator L is defined by

$$L = \partial^2 - u, \tag{5.6}$$

while the pseudodifferential operators X and E can be read off the matrix $\mathbb{V}$ and are

$$X = \xi(v) = -\partial^{-1} v - \partial^{-2}(\tfrac{1}{2} v_x), \tag{5.7}$$

$$E = \psi(\phi(v)) = \tfrac{1}{2} v_x - v\partial. \tag{5.8}$$

The relations $E = (XL)_+$ and the commutativity of the diagram (4.30) are easily checked.

The next example is the Boussinesq hierarchy, which is associated with the loop algebra over $\mathfrak{sl}(3)$. Again we simply have to compute the elements of Γ_Q, where the transversal submanifold Q is given by

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ u_0 & u_1 & 0 \end{pmatrix}. \tag{5.9}$$

Given the 1-form $v = (v_0, v_1)$ in $\mathcal{X}^*(\mathcal{N})$, the matrix $\mathbb{V} = \phi(v)$ is uniquely characterized by conditions (3.16) and (3.18), together with the obvious zero trace relation, to be

$$\mathbb{V} = \begin{pmatrix} \frac{2}{3} v_{0xx} - \frac{2}{3} v_0 u_1 - v_{1x} & v_1 - v_{0x} & v_0 \\ \begin{matrix} \frac{2}{3} v_{0xxx} - \frac{2}{3} v_{0x} u_1 - v_{1xx} + \\ (u_0 - \frac{2}{3} u_{1x} + \lambda) v_0 \end{matrix} & -\frac{1}{3} v_{0xx} + \frac{1}{3} v_0 u_1 & v_1 \\ \mathbb{V}_0^2 & \mathbb{V}_1^2 & -\frac{1}{3} v_{0xx} + \frac{1}{3} v_0 u_1 + v_{1x} \end{pmatrix}, \tag{5.10}$$

where

$$\begin{aligned} \mathbb{V}_0^2 &= -v_{1xxx} + (u_0 - \tfrac{4}{3} u_{1x} + \lambda) v_{0x} + (u_{0x} - \tfrac{2}{3} u_{1xx}) v_0 \\ &\quad - \tfrac{2}{3} v_{0xx} u_1 + (u_0 + \lambda) v_1 + \tfrac{2}{3} v_{0xxxx}, \\ \mathbb{V}_1^2 &= -v_{1xx} + \tfrac{1}{3} v_{0xxx} - \tfrac{1}{3} v_{0x} u_1 + (u_0 - \tfrac{1}{3} u_{1x} + \lambda) v_0 + v_1 u_1. \end{aligned} \tag{5.11}$$

The reduced Poisson pencil is

$$\begin{aligned} \dot{u}_0 &= \tfrac{2}{3} v_{0xxxxx} - \tfrac{4}{3} u_1 v_{0xxx} - 2u_{1x} v_{0xx} + (\tfrac{2}{3} u_1^2 + 2u_{0x} - 2u_{1xx}) v_{0x} \\ &\quad + (-\tfrac{2}{3} u_{1xxx} + \tfrac{2}{3} u_1 u_{1x} + u_{0xx}) v_0 \\ &\quad - v_{1xxxx} + u_1 v_{1xx} + 3u_0 v_{1x} + u_{1x} v_1 + 3\lambda\, v_{1x}, \\ \dot{u}_1 &= v_{0xxxx} - u_1 v_{0xx} + (3u_0 - 2u_{1x}) v_{0x} + (2u_{0x} - u_{1xx}) v_0 \\ &\quad - 2v_{1xxx} + 2u_1 v_{1x} + u_{1x} v_1 + 3\lambda\, v_{0x}. \end{aligned} \tag{5.12}$$

In this case the operators L, X and E are

$$\begin{aligned} L &= \partial^3 - u_1 \partial - u_0 \\ X &= -\partial^{-1} v_0 - \partial^{-2} v_1 - \partial^{-3}(v_{1x} - \tfrac{1}{3} v_{0xx} + \tfrac{1}{3} v_0 u_1) \\ E &= -(\tfrac{2}{3} v_{0xx} - \tfrac{2}{3} v_0 u_1 - v_{1x}) - (v_1 - v_{0x})\partial - v_0 \partial^2. \end{aligned} \tag{5.13}$$

Acknowledgements

The work is supported by the Italian MURST and by the GNFM of the Italian CNR.

References

[1] L.R. Abraham and J.E. Marsden, *Foundations of Mechanics*, 2nd Ed. (Addison-Wesley, Singapore 1978).

[2] M. Adler, On a trace functional for formal pseudo-differential operators and the symplectic structure for the KdV equations, Invent. Math. 50 (1979) 219–248.

[3] I. Bakas, The structure of the W_∞-algebra, Comm. Math. Phys. 123 (1989) 627–640.

[4] M. Bershadsky, Conformal field theories via hamiltonian reductions, Comm. Math. Phys. 139 (1991) 71–82.

[5] P. Bouwknegt and K. Schoutens eds., *W-symmetry*, Adv. Series in Math. Phys., Vol. XX (World Scientific, Singapore, 1995).

[6] N.J. Burroughs, M.F. DeGroot, T.J. Hollowood and J.L. Miramontes, Generalized Drinfel'd–Sokolov Hierarchies. II. The hamiltonian structures, Comm. Math. Phys. 153 (1993) 187–215.

[7] P. Casati, G. Falqui, F. Magri and M. Pedroni, The KP theory revisited. II. The reduction theory and the Adler–Gel'fand–Dickey brackets, reprint SISSA/3/96/FM (1996).

[8] P. Casati, G. Falqui, F. Magri and M. Pedroni, The KP theory revisited. III. The bihamiltonian action and Gel'fand–Dickey equations, reprint SISSA/4/96/FM (1996).

[9] P. Casati, G. Falqui, F. Magri and M. Pedroni, A Note on Fractional KdV Hierarchies, reprint SISSA/94/96/FM 1996, solv-int/9606010, J. Math. Phys., to appear.

[10] P. Casati, F. Magri and M. Pedroni, Bihamiltonian manifolds and τ-function, in: *Mathematical Aspects of Classical Field Theory 1991*, eds. M.J. Gotay et al., Contemporary Mathematics, Vol. 132 (American Mathematical Society, Providence, RI, 1992) pp. 213–234.

[11] P. Casati and M. Pedroni, Drinfeld–Sokolov reduction on a simple lie algebra from the bihamiltonian point of view, Lett. Math. Phys. 25 (1992) 89–101.

[12] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, Transformation groups for soliton equations, in: *Proc. of RIMS Symp. on Nonlinear Integrable Systems – Classical Theory and Quantum Theory*, eds. M. Jimbo and T. Miwa (World Scientific, Singapore, 1983) pp. 39–119.

[13] P. Di Francesco, C. Itzykson and J.-B. Zuber, Classical W-Algebras Comm. Math. Phys. 140 (1991) 543–567.

[14] L.A. Dickey, *Soliton equations and hamiltonian systems*, Adv. Series in Math. Phys, Vol. 12 (World Scientific, Singapore 1991).

[15] L.A. Dickey, Lectures on Classical W-Algebras, Notes of a course held in Cortona (Italy) (1993).

[16] I. Dorfman, *Dirac Structures and Integrability of Nonlinear Evolution Equations* (Wiley, New York, 1993).

[17] V.G. Drinfeld and V.V. Sokolov, Lie algebras and equations of Korteweg–de vries type, J. Sov. Math. 30 (1985) 1975–2036.

[18] V.A. Fateev and S.L. Luk'yanov, Additional symmetries and exactly-soluble models in two-dimensional conformal field theory, Sov. Sci. Rev. A. Phys. 15 (1990) 1–124.

[19] L. Fehér, L. O'Raifertaigh, P. Ruelle and A. Wipf, On Hamiltonian reductions of the WZNW theories, Phys. Rep. 222 (1992) 1–64.

[20] B.L. Feigin and E.V. Frenkel, Quantization of the Drinfel'd–Sokolov Reduction, Phys. Lett. 246 (1990) 75–83.

[21] B.L. Feigin and E.V. Frenkel, Affine Kac–Moody algebras at the critical level and Gel'fand–Dikii algebras, Internat. J. Modern Phys. A 7 (Supplement 1 A) (1992) 197–215.

[22] J.M. Figueroa-O'Farrill and E. Ramos, The algebra of differential operators on the circle and $W_{\mathrm{KP}}^{(q)}$ 27 (1993) 223–233.

[23] L. Frappat, E. Ragoucy and P. Sorba, W-algebras and superalgebras from constrained WZW models: A group theoretic classification 157 (1993) 499–531.

[24] I.M. Gelfand and L.A. Dickey, Fractional powers of operators and hamiltonian systems, Funct. Anal. Appl. 10 (1976) 259–273.

[25] T.G. Khovanova, The Gel'fand–Dikii algebras and the Virasoro algebra, Funct. Anat. Appl. 20 (1987) 332–334.

[26] P. Libermann and C.M. Marle, *Symplectic Geometry and Analytical Mechanics* (Reidel, Dordrecht, 1987).

[27] S.L. Luk'yanov, Quantization of the Gel'fand–Dickey brackets, Funct. Anal. Appl. 22 (1988) 255–262.

[28] F. Magri and C. Morosi, A characterization of the integrable systems through the theory of Poisson–Nijenhuis manifolds, Quaderni dell'Università di Milano, S 19/1984 (1984).
[29] Yu.I. Manin, Algebraic aspects of non-linear differential equations, J. Sov. Math. 11 (1979) 1–122.
[30] J.E. Marsden and T. Ratiu, Reduction of Poisson Manifolds, Lett. Math. Phys. 11 (1986) 161–169.
[31] C. Morosi and L. Pizzocchero, On the bihamiltonian interpretation of the lax formalism, Rev. Math. Phys. 7 (1995) 389–430.
[32] M. Pedroni, Equivalence of the Drinfel'd–Sokolov reduction to a bihamiltonian reduction, Lett. Math. Phys. 35 (1995) 291–302.
[33] O. Radul, Lie algebras of differential operators, their central extensions, and W-algebras, Funct. Anal. Appl. 25 (1991) 25–39.
[34] A.B. Zamolodchikov, Infinite additional symmetries in two dimensional quantum field theory, Theor. Math. Phys. 63 (1985) 1205–1210.